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Research Note

# On functional dependencies in q-Horn theories ☆

Toshihide Ibaraki [a], Alexander Kogan [b,c,*], Kazuhisa Makino [d]

[a] *Department of Applied Mathematics and Physics, Graduate School of Informatics, Kyoto University, Kyoto, Japan 606-8501*

[b] *Department of Accounting and Information Systems, Faculty of Management, Rutgers University, Newark, NJ 07102, USA*

[c] *RUTCOR, Rutgers University, 640 Bartholomew Road, Piscataway, NJ 08854-8003, USA*

[d] *Division of Systems Science, Graduate School of Engineering Science, Osaka University, Toyonaka, Osaka, Japan 560-8531*

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**Abstract**

This paper studies functional dependencies in q-Horn theories, and discusses their use in knowledge condensation. We introduce compact model-based representations of q-Horn theories, analyze the structure of functional dependencies in q-Horn theories, and show that every minimal functional dependency in a q-Horn theory $\Sigma$ can be expressed either by a single term or by a single clause. We also prove that the set of all minimal functional dependencies in $\Sigma$ is quasi-acyclic. We then develop polynomial time algorithms for recognizing whether a given functional dependency holds in a q-Horn theory, which is represented either by a q-Horn CNF or as the q-Horn envelope of a set of models. Finally, we show that every q-Horn theory has a unique condensation, and can be condensed in polynomial time. 

*Keywords:* Knowledge representation; q-Horn theory; Functional dependency; Condensation; Computational complexity; Conjunctive normal form

## 1. Introduction

The concept of functional dependencies was introduced in the theory of relational databases (see [1,11]) in the seventies, and ever since it has been commonly used in

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* Corresponding author.

*E-mail addresses:* ibaraki@i.kyoto-u.ac.jp (T. Ibaraki), kogan@rutcor.rutgers.edu (A. Kogan), makino@sys.es.osaka-u.ac.jp (K. Makino).

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logical database design (see, e.g., [30–32]). In [18], this concept was applied to the field of knowledge representation. A functional dependency states that, in all models of a theory, the value of a variable is a function of values of some other variables. Thus, the existence of a functional dependency is an important property of the theory.

If a functional dependency holds in a theory, then it may be possible to simplify the theory by eliminating the variable whose value is determined by the values of other variables. The repeated application of this procedure will result in the so-called "condensed" theory, which does not have any functional dependencies, may have much fewer variables than the original theory, and may be structurally simpler than the original theory (see [18]). Although, in general, the condensed theory can be longer than the original theory, have more involved structure, and hence may be harder to reason with, it is shown in this paper that the condensation of q-Horn theories results (in the worst case) in only a moderate polynomial increase in the length of a q-Horn theory, preserves the q-Horn structure of the theory, and can potentially reduce the length of the original theory significantly. Moreover, the condensed theory can be viewed as the "core" of the original theory, and thus condensation can reveal important structural information about the problem domain.

The computational expense of condensing a theory is incurred at the preprocessing stage. However, the use of the condensed theory can result in a significant speedup in answering queries to the knowledge base, and therefore the expense of condensation can be quickly amortized over the regular use of the knowledge base. As discussed in [18], condensation has important connections with other types of knowledge preprocessing such as knowledge compilation (see [20,21,37]) and knowledge compression (see [5,13–16]).

Since Horn clauses are one of the most important knowledge representation languages, the main effort in [18] was devoted to the studies of functional dependencies in Horn theories. A characterization of the combinatorial structure of such functional dependencies was obtained, and efficient polynomial algorithms for recognizing them were developed. It was also shown that any functional dependency in a Horn theory can be expressed by a single positive term, any Horn theory has a unique condensation, and can be condensed in polynomial time.

In this paper, we investigate functional dependencies in q-Horn theories. The concept of a q-Horn conjunctive normal form (CNF) was introduced in [3] as a natural generalization of quadratic and (disguised) Horn CNFs. It is known [3] that the satisfiability problem (SAT) for a q-Horn CNF can be solved in linear time, and [6] that the problem of recognizing whether a CNF is q-Horn can also be solved in linear time. The class of q-Horn CNFs not only includes the most important computationally tractable classes of CNFs such as Horn and quadratic, but it also extends significantly the universe of CNFs for which both SAT and recognition are easy computational problems [4]. Therefore, q-Horn theories have a great potential in practical applications, including knowledge based systems, and testing and verification in computer engineering [3,6]. The theoretical significance of the class of q-Horn CNFs was demonstrated in [4], where a polynomial time SAT algorithm was developed for a large class of CNFs which are structurally similar to q-Horn.

While knowledge representation traditionally focused on the CNF representations of theories, recently significant progress has been made in studying compact model-based representations of theories (such as the Horn envelope [22]) and developing reasoning with models (see [8,20,23–25]). Therefore, in addition to q-Horn CNFs, we introduce the concept of q-Horn envelope $QH(\Sigma)$ of a set of models $\Sigma$ for representing a special canonical type of q-Horn theories which have a $QH$-partition.

To analyze the structure of functional dependencies in a q-Horn theory, we first develop an analytical expression for a functional dependency in a general Boolean theory through prime implicants (implicates) of the theory. We then show that the properties of functional dependencies in a q-Horn theory $\Sigma$ are similar to those in a Horn theory. Namely, we show that every minimal functional dependency can be represented either by a single term or by a single clause, and furthermore, the set of variables can be partitioned into two disjoint subsets $Q$ and $H$ such that every minimal functional dependency in $\Sigma$ involves variables either only from $Q$ or only from $H$, with the former ones all being simple, and the latter ones all being functional dependencies in the disguised Horn theory $\Sigma[H]$. We also prove that the set of all minimal functional dependencies is quasi-acyclic. We then provide a polynomial time algorithm for checking whether a given functional dependency holds in a q-Horn theory, which is represented either by a q-Horn CNF, or as the q-Horn envelope of a set of models. Finally, we use these results to show that any q-Horn theory has a unique condensation and can be condensed in polynomial time.

## 2. Notation and basic concepts

Propositional variables taking the values in $\{0, 1\}$ (meaning *false* and *true* respectively, and assuming $0 < 1$) will be denoted by lower case Latin letters (usually from the end of the alphabet), with $\overline{x}$ denoting the negation of $x$. Propositional variables and their negations will be called *literals*, with the variables themselves called *positive literals* and their negations called *negative literals*. Upper case Latin letters (usually from the end of the alphabet) will be used to denote sets of propositional variables, with the letter $V$ reserved to denote the set of all variables (in most cases $V = \{x_1, x_2, \ldots, x_n\}$). Boolean vectors (points, or models) in $\{0, 1\}^n$ will be denoted by lower case Greek letters, with $\alpha[X]$ denoting the restriction of a point $\alpha \in \{0, 1\}^n$ to the set of variables in $X \subseteq V$.

A set of Boolean vectors (also called *models*) $\sigma \subseteq \{0, 1\}^n$ is called a *theory* (or a *Boolean function* $\{0, 1\}^n \to \{0, 1\}$, identified with its set of *true* points, i.e., the points assigned the value 1). We will denote by $\Sigma[X]$ the theory $\Sigma$ restricted to the variables in $X$. The number of models of a theory $\Sigma$ will be denoted by $|\Sigma|$.

We shall call a disjunction (conjunction) of literals a *clause* (*term*), and in many cases will not distinguish between a clause and the set of literals it contains. A clause $C'$ is said to *subsume* a clause $C$ if $C$ contains all the literals in $C'$. It is well known that any theory can be represented as a conjunction of clauses called *conjunctive normal form* (CNF). In some cases, we will not make a distinction between a CNF and the theory it represents. The *length* of a CNF $\mathcal{F}$ (i.e., the number of literals in it) will be denoted by $|\mathcal{F}|$.

A clause $C$ is called an *implicate* of a theory $\Sigma$ if its set of models contains $\Sigma$, and this will be denoted as $\Sigma \models C$. Clearly, each clause of a CNF $\mathcal{F}$ is an implicate of the theory represented by $\mathcal{F}$. A clause $C$ is called a *prime* implicate of a theory $\Sigma$ if $C$ is an implicate of $\Sigma$, and $\Sigma$ does not have a distinct implicate $C'$ that subsumes $C$. A CNF consisting only of prime implicates of the theory it represents is called *prime*. Every theory can be represented by the conjunction of all its prime implicates. Using Boolean duality, one can similarly define the concepts of *disjunctive normal form* (DNF), *implicant*, *prime implicant*, etc.

A clause containing a single literal is called a *unit* clause, while a clause containing two literals will be called *quadratic*. As argued in [18], without loss of generality, we can assume that all theories considered here have no unit implicates, and therefore all their quadratic implicates are prime. A CNF is called *quadratic* if it contains only quadratic clauses. It is well known [2] that the *satisfiability problem* (SAT) for a quadratic CNF can be solved in linear time, where SAT is the problem of checking whether the theory represented by a given CNF contains at least one model. A theory is called *quadratic* if it can be represented by a quadratic CNF. Note that every prime implicate of a quadratic theory is quadratic, and therefore every prime CNF of a quadratic theory is quadratic.

A clause is called *Horn* if it contains at most one positive literal (see [17]). A CNF is called *Horn* if it contains only Horn clauses. A very important property of Horn CNFs is the linear time complexity of SAT (see [9,34]). A theory is called *Horn* if there exists a Horn CNF representing it. It is known (see [13,14]) that every prime implicate of a Horn theory is Horn, and therefore any prime CNF of a Horn theory is Horn.

## 3. q-Horn theories

For a CNF $\mathcal{F}$ and a subset $W \subseteq V = \{x_1, x_2, \ldots, x_n\}$, the CNF $\mathcal{F}_W$, which is obtained from $\mathcal{F}$ by complementing all occurrences of the variables in $W$, will be called the *renaming* of $\mathcal{F}$ with respect to $W$. For example, if $\mathcal{F} = (\bar{x}_1 \vee x_2)(x_1 \vee x_3 \vee \bar{x}_4)(x_2 \vee \bar{x}_3 \vee \bar{x}_4)$ and $W = \{x_1, x_2\}$, then $\mathcal{F}_W = (x_1 \vee \bar{x}_2)(\bar{x}_1 \vee x_3 \vee \bar{x}_4)(\bar{x}_2 \vee \bar{x}_3 \vee \bar{x}_4)$. We say that $\mathcal{F}$ and $\mathcal{F}'$ are *congruent* if $\mathcal{F}' = \mathcal{F}_W$ for some $W \subseteq V$. A CNF $\mathcal{F}$ is called *disguised* (or *renamable*) *Horn* [29] if $\mathcal{F}$ is congruent to a Horn CNF. It is known [7,29] that both SAT and recognition problems for a disguised Horn CNF can be solved in linear time. A theory is called *disguised Horn* if it can be represented by a disguised Horn CNF.

We shall now define a q-Horn CNF. Let $\mathcal{F}$ be a CNF given by

$$\mathcal{F} = \bigwedge_{k=1}^{m} C_k, \tag{1}$$

where

$$C_k = \bigvee_{x_i \in P(C_k)} x_i \vee \bigvee_{x_i \in N(C_k)} \bar{x}_i, \tag{2}$$

with $P(C_k), N(C_k) \subseteq V$ such that $P(C_k) \cap N(C_k) = \emptyset$ for all $k$. Note that $\mathcal{F}$ is *quadratic* (respectively, Horn) if $|P(C_k) \cup N(C_k)| \leqslant 2$ (respectively, $|P(C_k)| \leqslant 1$) for all $k$. For a CNF $\mathcal{F}$ of (1), let us define the following linear programming problem LP($\mathcal{F}$) over the set of variables $\phi(x_1), \phi(x_2), \ldots, \phi(x_n)$:

$$\begin{aligned}&\text{minimize } Z\\&\text{subject to } \sum_{x_i\in P(C_k)}\phi(x_i)+\sum_{x_i\in N(C_k)}\bigl(1-\phi(x_i)\bigr)\leqslant Z \quad \text{for all } k=1,2,\ldots,m, \text{ and}\\&\qquad 0\leqslant \phi(x_i)\leqslant 1 \quad \text{for all } i=1,2,\ldots,n.\end{aligned} \tag{3}$$

We denote the optimal value of (3) as $Z(\mathcal{F})$. A CNF $\mathcal{F}$ is called *q-Horn* if $Z(\mathcal{F})\leqslant 1$ (see [3]). It is known [6] that the problem of recognizing whether a CNF is q-Horn can be solved in linear time, and [3,6] that SAT for a q-Horn CNF also can be solved in linear time. The linear time complexity of the recognition problem is based on the structure of the *QH*-partition described below. Once a desired *QH*-partition (with a renaming) is found for a q-Horn CNF, SAT can be solved in linear time by first solving Horn SAT for the $H$-part and then solving 2-SAT for the $Q$-part (see [3] for details).

Let $\mathcal{F}$ be a CNF on the variable set $V$. A partition $\{Q,H\}$ of $V$ (i.e., $Q\cup H=V$ and $Q\cap H=\emptyset$) is called a *QH-partition* for $\mathcal{F}$ if every clause $C$ in $\mathcal{F}$ satisfies the following three conditions:

(i) $C$ contains not more than two variables from $Q$.
(ii) $C$ contains at most one positive literal from $H$. (4)
(iii) If $C$ contains a positive literal from $H$, then it contains no variable from $Q$.

For example, $\mathcal{F}=(\bar{x}_1\vee x_2\vee\bar{x}_4\vee\bar{x}_5)(x_1\vee x_3\vee\bar{x}_5\vee\bar{x}_6)(\bar{x}_4\vee x_5\vee\bar{x}_6)$ has a *QH*-partition, where $Q=\{x_1,x_2,x_3\}$ and $H=\{x_4,x_5,x_6\}$. If $\mathcal{F}$ has a *QH*-partition, then by assigning $\phi(x_i)=1/2$ for all $x_i\in Q$ and $\phi(x_i)=1$ for all $x_i\in H$, we have $Z(\mathcal{F})\leqslant 1$, showing that $\mathcal{F}$ is q-Horn. In general, the following criterion holds.

**Lemma 3.1** [3]. *A CNF is q-Horn if and only if it is congruent to a CNF having a QH-partition.*

Lemma 3.1 shows that the class of q-Horn CNFs includes as special cases both quadratic ($H=\emptyset$) and disguised Horn ($Q=\emptyset$) CNFs. A q-Horn CNF can be congruent to several distinct CNFs having *QH*-partitions. For example, if the variable $x_3$ is renamed in the CNF above, then the resulting CNF is Horn, and therefore is q-Horn having a *QH*-partition, where $Q=\emptyset$ and $H=\{x_1,x_2,x_3,x_4,x_5,x_6\}$. This shows that the CNF above is in fact disguised Horn. Note that there are q-Horn CNFs which are neither quadratic nor disguised Horn; e.g.,

$$\mathcal{F}=(\bar{x}_1\vee\bar{x}_2\vee x_3)(\bar{x}_2\vee\bar{x}_3\vee x_1)(\bar{x}_3\vee\bar{x}_1\vee x_2)(\bar{x}_1\vee\bar{x}_4\vee\bar{x}_5)(\bar{x}_2\vee\bar{x}_4\vee x_5)(\bar{x}_3\vee\bar{x}_5\vee x_4)(\bar{x}_6\vee x_4\vee x_5).$$

This prime CNF has a *QH*-partition with $Q=\{x_4,x_5\}$ and $H=\{x_1,x_2,x_3,x_6\}$, and is neither quadratic nor disguised Horn.

As shown in [6], there is a linear time algorithm that checks whether a given CNF is q-Horn, and if yes, produces a partition $\{Q,H\}$ and a renaming such that the renamed CNF has the *QH*-partition. This partition is optimal in the sense that if there is a CNF having a $Q'H'$-partition which is congruent to the given CNF, then $Q'\supseteq Q$.

**Lemma 3.2** [3]. *If a theory $\Sigma$ can be represented by a CNF having a QH-partition, then every prime implicate of $\Sigma$ satisfies the three conditions of* (4).

We shall call a theory *q-Horn* if it can be represented by a q-Horn CNF. We shall say that a q-Horn theory has a *QH*-partition if it can be represented by a CNF having a *QH*-partition.

## 4. Model-based representations of q-Horn theories

If a theory $\Sigma$ is represented by the set of all its models, then one can construct in polynomial time a prime CNF representing $\Sigma$ [40]. Hence, the results in the previous section imply that one can check in polynomial time whether $\Sigma$ is q-Horn, and if yes, construct a partition $\{Q, H\}$ and a renaming that transforms $\Sigma$ into a theory having the *QH*-partition. However, the representation of a theory by the set of all its models is exceedingly long. Therefore, it is common in reasoning with models [20,23,25] to represent a theory using a small subset of its models, which is possible if the models satisfy a certain closure property. It is well known (see [8,33]) that a Horn theory can be characterized by the following property of its models.

**Theorem 4.1** [8,33]. *A theory $\Sigma$ is Horn if and only if $\alpha, \beta \in \Sigma$ implies $\alpha \wedge \beta \in \Sigma$.*

Here the point $\gamma = \alpha \wedge \beta$ is defined by $\gamma_i = \alpha_i \wedge \beta_i$, $i = 1, 2, \ldots, n$, and is called the intersection of $\alpha$ and $\beta$. This property leads to a way of representing a Horn theory by a subset of its models, which has the property that all the other models can be obtained as intersections of some models in the subset. The smallest such subset is called the set of *characteristic models* [20,23,25]. For an arbitrary theory $\Sigma$, its intersection closure is called the *Horn envelope* of $\Sigma$ and is denoted by $H(\Sigma)$ (see [22]). Clearly, $H(\Sigma)$ is the minimum Horn superset of $\Sigma$; i.e., for any Horn theory $\Sigma' \supseteq \Sigma$, it holds that $H(\Sigma) \subseteq \Sigma'$.

It was recently shown in [10] that quadratic theories also have a semantic definition.

**Theorem 4.2** [10]. *A theory $\Sigma$ is quadratic if and only if $\alpha, \beta, \gamma \in \Sigma$ implies $(\alpha \wedge \beta) \vee (\beta \wedge \gamma) \vee (\gamma \wedge \alpha) \in \Sigma$.*

Here the point $\gamma = \alpha \vee \beta$ is defined by $\gamma_i = \alpha_i \vee \beta_i$, $i = 1, 2, \ldots, n$, and it is called the union of $\alpha$ and $\beta$. This result allows to define the quadratic envelope of an arbitrary theory $\Sigma$ (denoted by $Q(\Sigma)$), and to use compact model-based representations of quadratic theories, i.e., to represent a quadratic theory by a subset of its models $\Sigma$ such that $Q(\Sigma)$ is the theory.

Unfortunately, no semantic definition is known for disguised Horn theories, and the results of [10] imply that it is unlikely to exist. Therefore, we shall restrict our attention to defining compact model-based representations for q-Horn theories that have a given *QH*-partition.

Given a partition $\{Q, H\}$ of the variable set $V$, let the *QH-convolution* $\alpha \star \beta \star \gamma$ of any three points $\alpha, \beta, \gamma \in \{0, 1\}^n$ denote the point defined by:

$$
\begin{aligned}
(\alpha \star \beta \star \gamma)[H] &= \alpha[H] \wedge \beta[H] \wedge \gamma[H], \\
(\alpha \star \beta \star \gamma)[Q] &= (\alpha[Q] \wedge \beta[Q]) \vee (\beta[Q] \wedge \gamma[Q]) \vee (\gamma[Q] \wedge \alpha[Q]).
\end{aligned}
$$

Note that $(\alpha \star \alpha \star \beta)[H] = \alpha[H] \wedge \beta[H]$ and $(\alpha \star \alpha \star \beta)[Q] = \alpha[Q]$ hold.

**Lemma 4.3.** *If $\Sigma$ is a q-Horn theory having a given QH-partition, then $\alpha, \beta, \gamma \in \Sigma$ implies $\alpha \star \beta \star \gamma \in \Sigma$.*

**Proof.** Let us consider a prime implicate $C$ of $\Sigma$. By Lemma 3.2, $C$ satisfies (4). If $C$ contains no literal from $Q$, then $C$ is a Horn clause on variables in $H$. Hence by Theorem 4.1, $C(\alpha \star \beta \star \gamma) = 1$. On the other hand, if $C$ contains a literal from $Q$, then $C$ contains no positive literal from $H$. If any of $\alpha$, $\beta$ and $\gamma$, say $\alpha$, satisfies $C$ by making a negative literal from $H$ true (i.e., $\alpha[H]$ satisfies $C$), then $\alpha[H] \wedge \beta[H] \wedge \gamma[H] \leqslant \alpha[H]$ implies that $(\alpha \star \beta \star \gamma)[H]$ satisfies $C$. Hence $C(\alpha \star \beta \star \gamma) = 1$. Otherwise, each of $\alpha$, $\beta$ and $\gamma$ satisfies $C$ by making only literals from $Q$ true. Then we can view $C$ as a quadratic clause on $Q$, and by Theorem 4.2, we have $C(\alpha \star \beta \star \gamma) = 1$. □

**Lemma 4.4.** *Given a theory $\Sigma$ and a partition $\{Q, H\}$ of the variable set $V$, if $\alpha, \beta, \gamma \in \Sigma$ implies $\alpha \star \beta \star \gamma \in \Sigma$, then $\Sigma$ is a q-Horn theory having the corresponding QH-partition.*

**Proof.** We shall prove this lemma by showing that every prime implicate

$$C = \bigvee_{x_i \in P(C)} x_i \vee \bigvee_{x_i \in N(C)} \bar{x}_i$$

of $\Sigma$ satisfies the conditions (i), (ii) and (iii) of (4).

Let us first prove (ii), i.e., that $|P(C) \cap H| \leqslant 1$. Let us assume that this condition does not hold, e.g., $x_k, x_m \in P(C) \cap H$. Since $C$ is a prime implicate, there exists a model $\alpha$ (respectively, $\beta$) in $\Sigma$ which satisfies $C$ by making only $x_k$ (respectively, $x_m$) true. Then $C(\alpha \star \alpha \star \beta) = 0$, a contradiction.

To prove (iii), assume that $P(C) \cap H = \{x_m\}$ and $C$ contains a literal $l_k$ from $Q$. Since $C$ is a prime implicate, there exists a model $\alpha$ (respectively, $\beta$) in $\Sigma$ which satisfies $C$ by making only $x_m$ (respectively, $l_k$) true. Let us then consider the model $\alpha \star \alpha \star \beta$, and let $H' = H \cap (P(C) \cup N(C))$. Then one can see that $(\alpha \star \alpha \star \beta)[H'] = \beta[H']$. Since $(\alpha \star \alpha \star \beta)[Q] = \alpha[Q]$, it then follows that $C(\alpha \star \alpha \star \beta) = 0$, a contradiction.

Let us finally consider (i). Let $Q' = Q \cap (P(C) \cup N(C))$ and assume that $|Q'| \geqslant 3$, say, the literals $l_k, l_m, l_h$ are in $Q'$. Then, since $C$ is a prime implicate, there exists a model $\alpha$ (respectively, $\beta$ and $\gamma$) in $\Sigma$ which satisfies $C$ by making only $l_k$ (respectively, $l_m$ and $l_h$) true. Then $\alpha[H] = \beta[H] = \gamma[H]$. Let us now consider the model $\alpha \star \beta \star \gamma$. Then the tautology $(l \wedge \bar{l}) \vee (\bar{l} \wedge \bar{l}) \vee (\bar{l} \wedge l) = \bar{l}$ for $l = l_k, l_m, l_h$ implies that $C(\alpha \star \beta \star \gamma) = 0$, a contradiction. □

Lemmas 4.3 and 4.4 imply the following semantic definition of q-Horn theories having a given *QH*-partition.

**Theorem 4.5.** *A theory $\Sigma$ is q-Horn having a given QH-partition if and only if $\alpha, \beta, \gamma \in \Sigma$ implies $\alpha \star \beta \star \gamma \in \Sigma$.*

Now we can define a compact model-based representation of q-Horn theories having a given *QH*-partition. Given a theory $\Sigma$ and a partition $\{Q, H\}$ of $V$, let $QH(\Sigma)$ be the closure of $\Sigma$ under the operation of $QH$-convolution. We shall call this closure $QH(\Sigma)$ the *q-Horn envelope* of $\Sigma$. If $\Sigma$ is a q-Horn theory having a $QH$-partition, then one can find a minimal subset of models $\mathrm{Char}(\Sigma)$ such that $QH(\mathrm{Char}(\Sigma)) = \Sigma$. However, in contrast with Horn theories, $\mathrm{Char}(\Sigma)$ may not be unique.

## 5. Functional dependencies

For a set of variables $X \subseteq V$ and a variable $y \in V \setminus X$, the expression $X \to y$ is called a *functional dependency* in a theory $\Sigma$ if the values of the variables in $X$ determine the value of $y$ in every model of $\Sigma$, i.e., for any $\alpha, \beta \in \Sigma$, $\alpha[X] = \beta[X]$ implies $\alpha[y] = \beta[y]$. Various properties of functional dependencies in general Boolean theories are discussed in [18].

A functional dependency $X \to y$ in $\Sigma$ is *minimal* if there is no $X' \subset X$ such that $X' \to y$ holds in $\Sigma$. A functional dependency $X \to y$ is *simple* if $|X| = 1$. In theories without unit implicates, any simple functional dependency is minimal. Moreover, if $x \to y$ holds in $\Sigma$, then $y \to x$ must also hold in $\Sigma$, because a Boolean function of a single variable, which is not a constant, can be either an identity ($y = x$), or its negation ($y = \overline{x}$). Therefore, a simple functional dependency $x \to y$ holds in $\Sigma$ if and only if $x$ and $y$ are either logically equivalent or logically opposite in $\Sigma$.

A functional dependency $X \to y$ in $\Sigma$ states that $y$ is a Boolean function $f$ of the variables in $X$, i.e., $\alpha[y] = f(\alpha[X])$ for any $\alpha \in \Sigma$. In this case, we say that $f$ *expresses* $X \to y$ in $\Sigma$. This $f$ is not unique if $\Sigma[X]$ is a proper subset of $\{0,1\}^{|X|}$. Let us now discuss how to derive an analytical representation of such function $f$.

Let $\Sigma$ be a theory on the variable set $V$, $X \subseteq V$, and $y \in V \setminus X$. Let $PI_{(X,y)}(\Sigma)$ (respectively, $PI_{(X,\bar{y})}(\Sigma)$) denote the set of all prime implicates $C = \bigvee_{x_i \in P(C)} x_i \vee \bigvee_{x_i \in N(C)} \bar{x}_i$ of $\Sigma$ such that $P(C) \cup N(C) \subseteq X \cup \{y\}$ and $y \in P(C)$ (respectively, $y \in N(C)$). Let us now define two Boolean functions $f_{(X,y)}$ and $\overline{f}_{(X,\bar{y})}$ by

$$f_{(X,y)} = \bigvee_{C \in PI_{(X,y)}(\Sigma)} \Biggl( \bigwedge_{x_i \in P(C) \setminus \{y\}} \bar{x}_i \wedge \bigwedge_{x_i \in N(C)} x_i \Biggr), \tag{5}$$

$$\overline{f}_{(X,\bar{y})} = \bigwedge_{C \in PI_{(X,\bar{y})}(\Sigma)} \Biggl( \bigvee_{x_i \in P(C)} x_i \vee \bigvee_{x_i \in N(C) \setminus \{y\}} \bar{x}_i \Biggr). \tag{6}$$

**Lemma 5.1.** *If $X \to y$, where $y \notin X$, is a functional dependency in a theory $\Sigma$, then for any $\alpha \in \Sigma$,*

$$\alpha[y] = f_{(X,y)}(\alpha[X]) = \overline{f}_{(X,\bar{y})}(\alpha[X]).$$

**Proof.** Let us prove the first equality. Let $f$ be the maximal Boolean function which expresses the functional dependency $X \to y$ in $\Sigma$; i.e., no $g \leqslant f$ expresses it, where we write $g \leqslant f$ if $g(\alpha) = 1$ always implies $f(\alpha) = 1$. Let us first show that $f_{(X,y)} \leqslant f$ holds. Let $C$ be a clause in $PI_{(X,y)}(\Sigma)$, and let $t_C = \bigwedge_{x_i \in P(C) \setminus \{y\}} \bar{x}_i \wedge \bigwedge_{x_i \in N(C)} x_i$. Then no model $\alpha$ in $\Sigma$ with $\alpha[y] = 0$ satisfies $t_C(\alpha) = 1$. Therefore, $y = f \vee t_C$ still expresses

$X \to y$. By the maximality of $f$, $t_C$ is an implicant of $f$ (in fact, $t_C$ is a prime implicant of $f$). Thus $f \geqslant f_{(X,y)}$ holds.

Let us assume now that $f$ is represented by the DNF $\mathcal{D}$ consisting of all prime implicants. Let us remove from $\mathcal{D}$ all terms $t$ such that $t(\alpha) = 0$ holds for all models $\alpha$ in $\Sigma$, and denote the resulting DNF by $\mathcal{D}^*$. Then $\mathcal{D}^*$ and $f$ have the same value on all models of $\Sigma$, and therefore $\mathcal{D}^*$ still expresses $X \to y$ in $\Sigma$. Since every term $t_C$ of $f_{(X,y)}$ remains in $\mathcal{D}^*$, we have $\mathcal{D}^* \geqslant f_{(X,y)}$.

To prove the inverse inequality $\mathcal{D}^* \leqslant f_{(X,y)}$, let us consider a term $t = \bigwedge_{l \in L(t)} l$ in $\mathcal{D}^*$, i.e, $t$ is a prime implicant of $f$ and there exists a model $\alpha$ such that $t(\alpha) = 1$. Then the clause $C = \bigvee_{l \in L(t)} \bar{l} \vee y$ must be a prime implicate of $\Sigma$. Indeed, $C$ is an implicate of $\Sigma$, since $t(\alpha) = 1$ implies $\alpha[y] = 1$ for every model $\alpha \in \Sigma$. Obviously, $C_y = \bigvee_{l \in L(t)} \bar{l}$ is not an implicate of $\Sigma$, since there exists $\alpha \in \Sigma$ such that $t(\alpha) = 1$. Additionally, for every $l' \in L$ the clause $C_{l'} = \bigvee_{l \in L(t) \setminus l'} \bar{l} \vee y$ is not an implicate of $\Sigma$, since otherwise $t_{l'} = \bigwedge_{l \in L(t) \setminus l'} l$ would be an implicant of $f$, and hence $t$ would not be prime. Therefore, every term $t$ of $\mathcal{D}^*$ is in $f_{(X,y)}$, and hence $\mathcal{D}^* \leqslant f_{(X,y)}$, thus completing the proof of the first equality.

The second equality can be proven in a similar way. □

Prompted by the importance of Horn knowledge-based systems, we investigated in [18] functional dependencies in Horn theories, and in particular showed that they have the following characterization through prime implicates of the theory.

**Theorem 5.2** [18]. *A functional dependency $X \to y$ is minimal in a Horn theory $\Sigma$ if and only if, for all $x \in X$, clauses $\bar{y} \vee x$ and $y \vee \bigvee_{x \in X} \bar{x}$ are prime implicates of $\Sigma$. Hence, every minimal functional dependency in $\Sigma$ can be expressed by a single positive term $y = \bigwedge_{x \in X} x$.*

Given a set of functional dependencies $\mathcal{M}$, let us associate with it a directed graph $G(\mathcal{M})$ [15,35] whose set of vertices is the set of variables $V$, and an oriented arc $x \to y$ is in $G(\mathcal{M})$ if and only if $\mathcal{M}$ contains a functional dependency $X \to y$ such that $x \in X$. A set of functional dependencies $\mathcal{M}$ is called *acyclic* if its graph $G(\mathcal{M})$ contains no directed cycles. A set of functional dependencies $\mathcal{M}$ is called *quasi-acyclic* if all the cycles in $G(\mathcal{M})$ are created only by simple functional dependencies in $\mathcal{M}$, or more formally, if every arc within each strong component of $G(\mathcal{M})$ corresponds only to a simple functional dependency in $\mathcal{M}$. Quasi-acyclicity was introduced in [15] for the purpose of optimal compression of Horn knowledge bases.

**Theorem 5.3** [18]. *The set of all minimal functional dependencies in a Horn theory is quasi-acyclic.*

## 6. Structure of functional dependencies in q-Horn theories

In this section we study the structure of minimal functional dependencies in q-Horn theories, and show that it is similar to the case of Horn theories. Since functional dependencies are invariant with respect to variable renaming, and since any q-Horn CNF is congruent to a q-Horn CNF having a *QH*-partition by Lemma 3.1, the following analysis of structure of functional dependencies in q-Horn theories will focus only on q-Horn theories with given *QH*-partitions.

**Lemma 6.1.** *If $\Sigma$ is a q-Horn theory having a QH-partition, then $\Sigma[H]$ is a Horn theory.*

**Proof.** If $\alpha, \beta \in \Sigma$, then $(\alpha \star \alpha \star \beta)[H]$ $(= \alpha[H] \wedge \beta[H]) \in \Sigma[H]$ by Theorem 4.5, proving that $\Sigma[H]$ is Horn by Theorem 4.1. □

Obviously, if $X \to y$ is a minimal functional dependency in $\Sigma[H]$, then it is a minimal functional dependency in $\Sigma$.

**Lemma 6.2.** *Let $\Sigma$ be a q-Horn theory having a QH-partition, and let $X \to y$ be a minimal functional dependency in $\Sigma$.*
(I) *If $y \in Q$, then $X = \{x_i\}$ holds for some $x_i \in Q$.*
(II) *If $y \in H$, then $X \subseteq H$ and $X \to y$ is a minimal functional dependency in $\Sigma[H]$.*

**Proof.** Let us first prove (I). By Lemma 5.1, the Boolean function $f_{(X,y)}$ expresses $X \to y$ in $\Sigma$. Every term

$$t_C = \bigwedge_{x_i \in P(C) \setminus \{y\}} \bar{x}_i \wedge \bigwedge_{x_i \in N(C)} x_i$$

in $f_{(X,y)}$ is constructed from a prime implicate

$$C = \bigvee_{x_i \in P(C)} x_i \vee \bigvee_{x_i \in N(C)} \bar{x}_i$$

of $\Sigma$ such that $P(C) \cup N(C) \subseteq X \cup \{y\}$. Since $y \in P(C) \cap Q$, Lemma 3.2 implies

$$P(C) \cap H = \emptyset \quad \text{and} \quad \big|\big(P(C) \cup N(C)\big) \cap Q\big| \leqslant 2. \tag{7}$$

Let $g$ be the function obtained from $f_{(X,y)}$ by assigning the value 0 to all variables $x_i \in H \cap X$. Then the conditions of (7) and the expression of (5) imply that the only terms, which will remain in $g$, are those for which $N(C) \cap H = \emptyset$. All such terms obviously correspond to quadratic prime implicates of $\Sigma$

$$z_i^{\sigma} \vee y, \tag{8}$$

where $z_i^{\sigma} \in \{z_i, \bar{z}_i\}$, and $z_i \in X \cap Q$. Denoting the set of such $z_i$ as $Z$, we have

$$g = \bigvee_{z_i \in Z} \overline{z_i^{\sigma}}. \tag{9}$$

The expression (9) of the functional dependency implies that for every $\alpha \in \Sigma$ such that $\alpha[X \cap H] = (0 \cdots 0)$, if $\bigvee_{z_i \in Z} \overline{z_i^{\sigma}}(\alpha) = 0$ then $\alpha[y] = 0$. Therefore, $\Sigma$ has the implicate

$$C' = \bigvee_{x_i \in X \cap H} x_i \vee \bigvee_{z_i \in Z} \overline{z_i^{\sigma}} \vee \bar{y}.$$

Let us consider a prime implicate $C^*$ of $\Sigma$ such that $C^* \leqslant C'$. Then this $C^*$ must satisfy

$$P(C^*) \cap H = \emptyset, \tag{10}$$

since otherwise, by Lemma 3.2, $C^*$ would have the form of $C^* = x_i$ for some $x_i \in X \cap H$, contradicting the assumption that $\Sigma$ has no unit implicates. Note that (10) and Lemma 3.2 imply that $C^*$ is a quadratic clause which has the form of either (a) $\overline{z_i^\sigma} \vee \overline{z_j^\sigma}$ for some $z_i, z_j \in Z$, or (b) $\bar{y} \vee \overline{z_i^\sigma}$ for some $z_i \in Z$.

Let us consider the case of (a). Then, in view of (8), $\Sigma$ has three implicates $\overline{z_i^\sigma} \vee \overline{z_j^\sigma}$, $z_i^\sigma \vee y$, and $z_j^\sigma \vee y$. It follows then that $y$ is an implicate of $\Sigma$, contradicting the assumption that $\Sigma$ has no unit implicates.

In the case of (b), in view of (8), $\Sigma$ has two implicates $z_i^\sigma \vee y$, and $\bar{y} \vee \overline{z_i^\sigma}$. This implies that $y$ and $z_i^\sigma$ are logically opposite in $\Sigma$, and therefore, $z_i \to y$ is a functional dependency in $\Sigma$ which can be expressed as $y = \overline{z_i^\sigma}$. The minimality of $X \to y$ implies $X = \{z_i\}$, and hence $X \to y$ is a simple functional dependency involving only variables in $Q$.

Let us now prove (II), and consider again $f_{(X,y)}$ expressing $X \to y$ in $\Sigma$. Every term $t_C = \bigwedge_{x_i \in P(C)\setminus\{y\}} \bar{x}_i \wedge \bigwedge_{x_i \in N(C)} x_i$ in $f_{(X,y)}$ is constructed from a prime implicate

$$C = \bigvee_{x_i \in P(C)} x_i \vee \bigvee_{x_i \in N(C)} \bar{x}_i$$

of $\Sigma$. Since $y \in P(C) \cap H$, Lemma 3.2 implies that

$$P(C) = \{y\} \quad \text{and} \quad N(C) \subseteq H$$

holds. This and the minimality of $X \to y$ imply $X \subseteq H$, and hence $X \to y$ is a minimal functional dependency in $\Sigma[H]$. □

**Theorem 6.3.** *If $\Sigma$ is a q-Horn theory, then every minimal functional dependency in $\Sigma$ can be expressed either by a single term or by a single clause. Furthermore, the set of variables can be partitioned into two disjoint subsets $Q$ and $H$ such that every minimal functional dependency in $\Sigma$ involves variables either only from $Q$ or only from $H$* (*the former ones all being simple*)*, and the set of all minimal functional dependencies in $\Sigma$ is quasi-acyclic.*

**Proof.** Let us first assume that $\Sigma$ has a given *QH*-partition. By Lemma 6.2(I), a minimal functional dependency $X \to y$ such that $y \in Q$ is expressed by $y = x_i^\sigma$, where $x_i \in Q$ and $x_i^\sigma \in \{x_i, \bar{x}_i\}$. By Lemma 6.2(II), a minimal functional dependency $X \to y$ such that $y \in H$ is also a minimal functional dependency in $\Sigma[H]$. Since $\Sigma[H]$ is Horn by Lemma 6.1, Theorem 5.2 implies that $X \to y$ can be expressed by a single positive term Boolean function, i.e., $y = \bigwedge_{x_i \in X} x_i$.

A general q-Horn theory $\Sigma$ can be transformed into a q-Horn theory with a *QH*-partition by an appropriate renaming (see Lemma 3.1). This means that the expression of a minimal functional dependency in $\Sigma$ is either a single term (which may not be positive) or a single clause, with the latter case occurring if $y$ is renamed (e.g., $\bar{y} = \bigwedge_{x \in X} x \Leftrightarrow y = \bigvee_{x \in X} \bar{x}$).

Theorem 5.3 and the discussion above imply that the set of all minimal functional dependencies in $\Sigma$ is quasi-acyclic. □

Theorem 6.3 shows in particular that variables from $Q$ and $H$ do not "interact" (i.e., are not present together) in minimal functional dependencies in a q-Horn theory. This contrasts with the fact that, as the last example in Section 3 shows, variables from $Q$ and $H$ do interact in prime implicates of a q-Horn theory.

## 7. Recognition of functional dependencies in q-Horn theories

The *recognition problem* for functional dependencies consists in checking whether a given functional dependency $X \to y$ holds in a given theory $\Sigma$. To formally specify this problem, we have to define how $\Sigma$ is represented. In our discussion below, we shall separately consider the cases when $\Sigma$ is given as a q-Horn CNF and as the q-Horn envelope of a set of models.

It was shown in [18] that in the case of general Boolean theories it is CoNP-complete to check whether a functional dependency $X \to y$ holds in the theory represented by a given CNF $\mathcal{F}$. It was also shown in [18] that the problem can be solved in polynomial time if $\mathcal{F}$ is Horn. We use below a similar technique for the q-Horn case.

**Theorem 7.1.** *Given a q-Horn CNF $\mathcal{F}$ and a functional dependency $X \to y$, it can be checked in* $\mathrm{O}(|\mathcal{F}|)$ [1] *time whether $X \to y$ holds in the theory $\Sigma$ represented by $\mathcal{F}$.*

**Proof.** Let us introduce a new variable $z_i'$ for every $z_i \in V \setminus (X \cup \{y\})$, and let us denote by $\mathcal{F}'$ the CNF obtained from $\mathcal{F}$ by substituting $\overline{y}$ for $y$ and $z_i'$ for $z_i$ for every $z_i \in V \setminus (X \cup \{y\})$. The CNF $\mathcal{F}'$ can be constructed in $\mathrm{O}(|\mathcal{F}|)$ time, since the membership $z \in X$ can be checked in $\mathrm{O}(1)$ time after spending $\mathrm{O}(|V|)$ preprocessing time. Note that $X \to y$ does not hold in $\Sigma$ if and only if there exist $\alpha, \beta \in \Sigma$ such that $\alpha[X] = \beta[X]$ and $\alpha[y] \neq \beta[y]$. Thus one can see that $X \to y$ does not hold in $\Sigma$ if and only if the CNF $\mathcal{F} \wedge \mathcal{F}'$ is satisfiable, i.e., there exists a solution to the following equation:

$$\mathcal{F} \wedge \mathcal{F}' = 1. \tag{11}$$

This satisfiability problem may not be q-Horn because of the substitution of $\overline{y}$ for $y$ in $\mathcal{F}'$. Let $\mathcal{F}_1$ and $\mathcal{F}_0$ be the CNFs obtained from (11) by substituting $y = 1$ and $y = 0$, respectively. Then $\mathcal{F}_1$ and $\mathcal{F}_0$ are q-Horn, and (11) has a solution if and only if at least one of $\mathcal{F}_1$ and $\mathcal{F}_0$ is satisfiable. Since $\mathcal{F}_1$ is obtained from $\mathcal{F}_0$ by exchanging $z_i$ and $z_i'$ for all $z_i \in V \setminus (X \cup \{y\})$, it is easy to see that $\mathcal{F}_1$ is satisfiable if and only if so is $\mathcal{F}_0$. Thus, the linear time algorithm for the q-Horn satisfiability problem (see [3]) can be employed to construct an $\mathrm{O}(|\mathcal{F}|)$ time algorithm for checking whether a functional dependency holds in the theory represented by a q-Horn CNF. □

**Corollary 7.2.** *Given a q-Horn CNF $\mathcal{F}$ and a functional dependency $X \to y$ in the theory represented by $\mathcal{F}$, it can be checked in* $\mathrm{O}(|X||\mathcal{F}|)$ *time whether $X \to y$ is minimal.*

**Proof.** The procedure consists in removing variables from $X$ one by one and using Theorem 7.1 to check whether the resulting functional dependency still holds. If $X \to y$ is

[1] In the following we use the notation $\phi = \mathrm{O}(\psi)$ to denote that there exists a constant $c$ such that $\phi \leqslant c\psi$.

not minimal, a minimal functional dependency $X' \to y$ with $X' \subset X$ will be produced as a by-product of this procedure. $\Box$

In the case of general Boolean theories, it was shown in [18] that if all the models of a theory $\Sigma$ are given, then one can check in $\mathrm{O}(|V||\Sigma|)$ time whether a functional dependency $X \to y$ holds in $\Sigma$. However, the number of models is typically enormous. Therefore, we shall focus on compact model-based representations, and consider the case when a q-Horn theory is given as the q-Horn envelope $QH(\Sigma)$ of a set of models $\Sigma$.

**Theorem 7.3.** *Given a set of models $\Sigma$, a partition $\{Q, H\}$ of the variable set $V$, and a functional dependency $X \to y$, it can be checked in $\mathrm{O}(|V||\Sigma|)$ time whether $X \to y$ holds in the q-Horn envelope $QH(\Sigma)$.*

**Proof.** First, let us consider the case $y \in Q$. By Lemma 6.2(I), $X \to y$ holds in $QH(\Sigma)$ if and only if there exists $x \in X \cap Q$ such that $x \to y$ holds in $QH(\Sigma)$. Let us show that, for $x, y \in Q$, $x \to y$ holds in $QH(\Sigma)$ if and only if it holds in $\Sigma$. The "only if" part is trivial. To prove the "if" part, assume that $x \to y$ holds in $\Sigma$, and take $\alpha, \beta, \gamma \in \Sigma$. Without loss of generality, consider the case $\alpha[x] = \beta[x]$. Then $x \to y$ implies $\alpha[y] = \beta[y]$. It now follows from the definition of $\star$ that $(\alpha \star \beta \star \gamma)[x] = \alpha[x]$ and $(\alpha \star \beta \star \gamma)[y] = \alpha[y]$, implying that $x \to y$ holds in $QH(\Sigma)$. Finally, checking whether there exists $x \in X \cap Q$ such that $x \to y$ holds in $\Sigma$ can be done in $\mathrm{O}(|V||\Sigma|)$ time.

Let us now consider the case $y \in H$. Lemma 6.2(II) implies that $X \to y$ holds in $QH(\Sigma)$ if and only if $X \cap H \to y$ holds in $QH(\Sigma)$. Since $(\alpha \star \alpha \star \beta)[H] = \alpha[H] \wedge \beta[H]$ and $(\alpha \star \beta \star \gamma)[H] = (\alpha[H] \wedge \beta[H]) \wedge \gamma[H]$, the set $QH(\Sigma)[H]$ is closed under intersection. By Theorem 4.1, we then have

$$QH(\Sigma)[H] = H(\Sigma[H]). \tag{12}$$

Therefore, it is sufficient to check that $X \cap H \to y$ holds in the Horn envelope $H(\Sigma[H])$. By [18, Theorem 3.5], this can be done in $\mathrm{O}(|H||\Sigma[H]|) = \mathrm{O}(|V||\Sigma|)$ time. $\Box$

**Corollary 7.4.** *Given a set of models $\Sigma$, a partition $\{Q, H\}$ of the variable set $V$, and a functional dependency $X \to y$, it can be checked in $\mathrm{O}(|V||\Sigma|)$ time whether $X \to y$ is minimal in the q-Horn envelope $QH(\Sigma)$.*

**Proof.** If $y \in Q$, then by Lemma 6.2(I), $X = \{x\}$ for some $x \in Q$, which can be checked in $\mathrm{O}(|Q|)$ time. It can obviously be checked in $\mathrm{O}(|\Sigma|)$ time whether $x \to y$ holds in $\Sigma$ (and therefore in $QH(\Sigma)$).

If $y \in H$, then Lemma 6.2(II) implies that $X \subseteq H$, which can be checked in $\mathrm{O}(|X| + |V|) = \mathrm{O}(|V|)$ time. Then by [18, Corollary 3.6], it can be checked in $\mathrm{O}(|V||\Sigma|)$ time whether $X \to y$ is minimal in the Horn envelope $H(\Sigma[H])$. $\Box$

## 8. Condensation of q-Horn theories

If a functional dependency $X \to y$ holds in a theory $\Sigma$, then one can *condense* $\Sigma$ by removing the variable $y$ from $\Sigma$. The resulting theory $\Sigma' = \Sigma[V \setminus y]$ together with

$y = f(X)$ can be used instead of $\Sigma$ in the knowledge-based system. If there exists a functional dependency that holds in $\Sigma'$, then this functional dependency can be used to condense $\Sigma'$ further. This *condensation process* can be repeated until the resulting theory (denoted by $\Sigma^c$) has no functional dependencies. Such theories will be called *condensed*.

The benefit of condensation is the reduction in the number of variables, which can make reasoning with the condensed theory simpler and faster. However, as was discussed in [18], in the case of general Boolean theories, condensation can be computationally disadvantageous because the expressions of functional dependencies and the structure of the condensed CNF can be complicated, and therefore computationally expensive to store and use. Also, the result of condensation can depend on the order in which functional dependencies are used in the process.

It was shown in [18] that the condensation of Horn theories does not present any of the problems mentioned above. Every functional dependency is expressed by a single positive term, the condensed theory is unique (up to the choice of representatives of logically equivalent variables), and every Horn theory (whether given by a Horn CNF or as a Horn envelope) can be condensed in polynomial time.

Since by Theorem 6.3 the structure of functional dependencies in q-Horn theories is not significantly more complicated than that of Horn theories, the condensation of q-Horn theories can be successfully accomplished as well. Note first that since every functional dependency in a q-Horn theory is expressed by a single term or a single clause, it is not computationally expensive to store and use.

Second, the condensation of any q-Horn theory remains q-Horn, thus preserving the linear time complexity of inference. In the case of the q-Horn envelope representation, the length of the representation is always decreased, since condensation simply removes columns from the matrix of models, and $QH(\Sigma^c) = QH(\Sigma)^c$ holds. In the case of the q-Horn CNF representation, the expressions of condensed variables should be substituted in the CNF. The same argument as in [18] shows that the resulting CNF can be rewritten as q-Horn. Although this transformation may result in lengthening the condensed CNF by a factor of $O(|V|^2)$, it can also lead to an exponential reduction in the length of the condensed CNF (see [18, Lemma 6.1 and Theorem 6.2]).

The proof of [18, Theorem 6.3] shows that if the set of minimal functional dependencies in a theory $\Sigma$ is quasi-acyclic, then the condensed theory $\Sigma^c$ does not depend on the order in which functional dependencies are used in condensation, and therefore, $\Sigma^c$ is uniquely defined (up to the choice of representatives of logically equivalent or logically opposite variables). By Theorem 6.3, this applies to q-Horn theories.

To condense a q-Horn theory, one has to find for every variable $y \in V$ a minimal functional dependency $X \to y$, if any exists. Obviously, such a minimal functional dependency exists if and only if the functional dependency $V \setminus y \to y$ holds in the given q-Horn theory. Theorems 7.1 and 7.3 and Corollaries 7.2 and 7.4 of this paper provide the bounds on the complexity of these computations for the q-Horn CNF and q-Horn envelope representations. Then, assuming without loss of generality that the given q-Horn theory has a *QH*-partition, we can proceed exactly as in the proof of [18, Theorem 6.4] to obtain the following result (in which $V^c$ denotes the set of variables of the condensed theory).

**Theorem 8.1.**

(1) *Given a theory $\Sigma$, the theory $\Sigma^c$ such that $QH(\Sigma^c) = QH(\Sigma)^c$, and the terms and clauses expressing all the variables in $V \setminus V^c$ through the variables in $V^c$, can be constructed in* $O(|V|^3|\Sigma|)$ *time.*

(2) *Given a q-Horn CNF $\mathcal{F}$, a q-Horn CNF representing the condensation of the theory represented by $\mathcal{F}$, and the terms and clauses expressing all the variables in $V \setminus V^c$ through the variables in $V^c$, can be constructed in* $O(|V|^2|\mathcal{F}|)$ *time.*

## 9. Concluding remarks

This paper studies functional dependencies in q-Horn theories, and discusses their use in knowledge condensation. First, we show how to express a functional dependency in a general Boolean theory using some prime implicants (implicates) of the theory. We then proceed to the discussion of compact model-based representations of q-Horn theories and introduce the concept of q-Horn envelope as a means of representing q-Horn theories having *QH*-partitions. Next, we analyze the structure of functional dependencies in q-Horn theories, and show that every minimal functional dependency in a q-Horn theory $\Sigma$ can be expressed either by a single term or by a single clause. Furthermore, the set of variables can be partitioned into two disjoint subsets $Q$ and $H$ such that every minimal functional dependency in $\Sigma$ involves variables either only from $Q$ or only from $H$, with the former ones all being simple, and the latter ones all being functional dependencies in the disguised Horn theory $\Sigma[H]$. We also prove that the set of all minimal functional dependencies in $\Sigma$ is quasi-acyclic. We then develop polynomial time algorithms for recognizing whether a given functional dependency holds in a q-Horn theory, which is represented either by a q-Horn CNF or as the q-Horn envelope of a set of models. Finally, we use the obtained results to show that every q-Horn theory has a unique condensation, and can be condensed in polynomial time.

In a forthcoming paper [19] we consider the problem of inferring all minimal functional dependencies in a q-Horn theory. We develop an incrementally polynomial time algorithm for generating all minimal functional dependencies, if a q-Horn theory is represented by a q-Horn CNF. On the other hand, if a q-Horn theory is given as a q-Horn envelope, we show that there exists a polynomial total time algorithm for generating all its minimal functional dependencies if and only if there exists a polynomial total time algorithm for dualizing a positive CNF.

Finally, we mention that a challenging direction of research is to extend the techniques and results of this paper to such generalizations of the q-Horn concept as monotone decomposition [38] and linear autarkies [28,39], and to the hierarchies of generalized Horn formulae [12,26,27,36].

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